Inegalități

1) Dem. că $(x+y)\cdot\left(\frac{1}{x}+\frac{1}{y}\right)\geq 4$, $(\forall)\ x;y\in\mathbb{R}^+$, $(\forall)\ z\in\mathbb{R}^+$

$$(x+y)\cdot\left(\frac{^{y)}1}{x}+\frac{^{x)}1}{y}\right)\geq 4 \iff (x+y)\cdot\left(\frac{y}{xy}+\frac{x}{xy}\right)\geq 4 \iff (x+y)\cdot\frac{x+y}{xy}\geq 4 \iff$$

$$\frac{(x+y)^2}{xy}\geq\frac{4}{1}\ \Big|\cdot xy \iff \frac{(x+y)^2}{\cancel{xy}}\cdot\cancel{xy}\geq 4\cdot xy \iff x^2+2xy+y^2\geq 4xy \iff$$

$$\iff x^2+2xy+y^2-4xy\geq 0 \iff x^2-2xy+y^2\geq 0 \iff (x-y)^2\geq 0\ (A)$$

$(\forall)\ x\in\mathbb{R}_+$, $(\forall)\ y\in\mathbb{R}_+ \implies (x+y)\cdot\left(\frac{1}{x}+\frac{1}{y}\right)\geq 4\ (A).$

2) $$\frac{(x+y+z)^2}{3}\geq xy+yz+zx \quad (*)$$

$$\frac{(x+y+z)^2}{3}\geq xy+yz+zx\ \Big|\cdot 3 \iff (x+y+z)^2\geq 3xy+3yz+3zx \iff$$

$$x^2+y^2+z^2+2xy+2yz+2xz-3xy-3yz-3zx\geq 0 \iff$$

$$x^2+y^2+z^2-xy-yz-xz\geq 0\ \Big|\cdot 2 \iff$$

$$2x^2+2y^2+2z^2-2xy-2yz-2xz\geq 0 \iff$$

$$x^2+x^2+y^2+y^2+z^2+z^2-2xy-2yz-2xz\geq 0 \iff$$

$$x^2-2xy+y^2+x^2-2xz+z^2+y^2-2yz+z^2\geq 0 \iff$$

$$(x-y)^2+(x-z)^2+(y-z)^2\geq 0$$

$(\forall)\ x;y;z\in\mathbb{R}_+ \implies \left.\begin{aligned}(x-y)^2&\geq 0\\(x-z)^2&\geq 0\\(y-z)^2&\geq 0\end{aligned}\right| \implies (x-y)^2+(x-z)^2+(y-z)^2\geq 0\ (A) \implies$

$\implies (*)\ (A).$

3) $x>0,\ y>0,\ x\cdot y=4 \Rightarrow (x+2)\cdot(y+2)\geq 16$

Știm că $\frac{a+b}{2}\geq\sqrt{ab}$, $(\forall)\, a\in\mathbb{R}_+,\ b\in\mathbb{R}_+ \Rightarrow$

$$\left.\begin{array}{l}\frac{x+2}{2}\geq\sqrt{x\cdot 2}\\ \frac{y+2}{2}\geq\sqrt{y\cdot 2}\end{array}\right\} \Rightarrow \frac{x+2}{2}\cdot\frac{y+2}{2}\geq\sqrt{x\cdot 2}\cdot\sqrt{y\cdot 2}\Rightarrow$$

$$\frac{(x+2)\cdot(y+2)}{4}\geq\sqrt{x\cdot y\cdot 4}\Rightarrow$$

$$\frac{(x+2)\cdot(y+2)}{4}\geq\sqrt{4\cdot 4}\Rightarrow\frac{(x+2)(y+2)}{4}\geq\frac{4}{1}\Rightarrow(x+2)(y+2)\cdot 1\geq 4\cdot 4\Rightarrow$$

$(x+2)(y+2)\geq 16$ (q.e.d.)

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4) Dem. că $(x^2+2x)\cdot(x^2+2x-4)+4\geq 0$, $(\forall)\, x\in\mathbb{R}$. $(*)$

Notăm $x^2+x=t\Rightarrow$

$(*)\Leftrightarrow t(t-4)+4\geq 0\Leftrightarrow t^2-2\cdot 2t+2^2\geq 0\Leftrightarrow$

$(t-2)^2\geq 0\Leftrightarrow(x^2+2x-2)^2\geq 0$ (A), $(\forall)\, x\in\mathbb{R}\Rightarrow$

$\Rightarrow (*)$ (A)

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5) Dem. că $(x^2-3x)\cdot(x^2-3x+1)+5\geq 0$, $(\forall)\, x\in\mathbb{R}$ $(*)$

$a^2\pm 2ab+b^2=(a\pm b)^2$

Notăm $x^2-3x=t\Rightarrow$

$(*)\Leftrightarrow t\cdot(t+1)+5\geq 0\Leftrightarrow t^2+t+5\geq 0$

$\Downarrow$

V1

$t^2+2\cdot t\cdot\frac{1}{2}+\frac{1}{4}+\frac{19}{4}\geq 0\Leftrightarrow$

$\left(t+\frac{1}{2}\right)^2+\frac{19}{4}\geq 0$ (A) deoarece

$\left(x^2-3x+\frac{1}{2}\right)^2\geq 0$, $(\forall)\, x\in\mathbb{R}$

$\frac{19}{4}\geq 0$

$\Downarrow$

$(*)$ (A)

V2 $t^2+t+5\geq 0$

$a=1\quad b=1\quad c=5$

$\Delta=b^2-4ac=1-4\cdot 5=-19$

$\Delta<0\Rightarrow t^2+t+5$ nu are soluții reale $\Rightarrow t^2+t+5>0$ (A)

$\Downarrow$

$(*)$ (A)

7) $\triangle ABC: AB = c,\ AC = b,\ BC = a$

$a^2 + b^2 + c^2 - ab - bc - ac = 0$

$\triangle ABC = \triangle$ echilateral

Am dem. că $a^2 + b^2 + c^2 - ab - bc - ac = 0 \Leftrightarrow$

$(a-b)^2 + (b-c)^2 + (a-c)^2 = 0$

$(\forall)\, a, b, c \in \mathbb{R} \Rightarrow \begin{array}{l} (a-b)^2 \geq 0 \\ (b-c)^2 \geq 0 \\ (a-c)^2 \geq 0 \end{array} \Rightarrow \begin{array}{c} a-b=0 \\ \text{și} \\ b-c=0 \\ \text{și} \\ a-c=0 \end{array} \Rightarrow$

$a = b,\ b = c,\ a = c \Rightarrow a = b = c \Rightarrow \triangle ABC = \triangle$ ech.

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