Inegalități: inegalitatea mediilor

Fie $\left.\begin{matrix} a>0 \\ b>0 \end{matrix}\right| \Rightarrow m_{aritmetică} = m_a = \dfrac{a+b}{2}$

$m_{geometrică} = m_g = \sqrt{ab}$

$m_{armonică} = m_h = \dfrac{2ab}{a+b}$

$a_1, a_2, \dots, a_n \in \mathbb{R}$

$m_a = \dfrac{a_1 + a_2 + \dots + a_n}{n}$

$a_1, a_2, \dots, a_n \in \mathbb{R}_+$

$m_h = \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}$

Dacă $\left.\begin{matrix} a>0 \\ b>0 \end{matrix}\right| \Rightarrow m_h \le m_g \le m_a \Rightarrow \dfrac{2ab}{a+b} \le \sqrt{ab} \le \dfrac{a+b}{2}$

Dem: $\dfrac{2ab}{a+b} \le \sqrt{ab} \Leftrightarrow \dfrac{2ab}{a+b} \le \dfrac{\sqrt{a}\cdot\sqrt{b}}{1} \Leftrightarrow 2ab \cdot 1 \le (a+b)\cdot\sqrt{a}\cdot\sqrt{b} \Leftrightarrow$

$(2ab)^2 \le (a+b)^2 \cdot (\sqrt{a}\cdot\sqrt{b})^2 \Leftrightarrow 4a^2b^2 \le (a+b)^2 \cdot a \cdot b \;\Big|\cdot \dfrac{1}{ab} \Leftrightarrow$

$4a^{\cancel{2}}b^{\cancel{2}} \cdot \dfrac{1}{\cancel{ab}} \le (a+b)^2 \cdot \cancel{ab} \cdot \dfrac{1}{\cancel{ab}} \Leftrightarrow 4ab \le (a+b)^2 \Leftrightarrow 4ab \le a^2 + 2ab + b^2 \Leftrightarrow$

$\Leftrightarrow 0 \le a^2 + 2ab + b^2 - 4ab \Leftrightarrow 0 \le a^2 - 2ab + b^2 \Leftrightarrow 0 \le (a-b)^2 \Leftrightarrow$

$\Leftrightarrow (a-b)^2 \ge 0$ (A), $(\forall)\, a>0$, $(\forall)\, b>0 \Rightarrow \dfrac{2ab}{a+b} \le \sqrt{ab}$ (A)

$\sqrt{ab} \le \dfrac{a+b}{2} \;\Big|\cdot 2 \Leftrightarrow 2\cdot\sqrt{a}\cdot\sqrt{b} \le \dfrac{a+b}{\cancel{2}}\cdot\cancel{2} \Leftrightarrow 2\sqrt{a}\cdot\sqrt{b} \le a+b \Leftrightarrow$

$\Leftrightarrow 0 \le a - 2\sqrt{a}\cdot\sqrt{b} + b \Leftrightarrow 0 \le (\sqrt{a})^2 - 2\sqrt{a}\cdot\sqrt{b} + (\sqrt{b})^2 \Leftrightarrow$

$0 \le (\sqrt{a} - \sqrt{b})^2 \Leftrightarrow (\sqrt{a} - \sqrt{b})^2 \ge 0$ (A), $(\forall)\, a>0$, $(\forall)\, b>0 \Rightarrow \sqrt{ab} \le \dfrac{a+b}{2}$

MINIME / MAXIME

Determinați valoarea minimă a expresiei: $(x\in\mathbb{R}, y\in\mathbb{R})$

$E(x) = x^2 + 9$

$(\forall)\, x\in\mathbb{R} \Rightarrow x^2 \ge 0$

Valoarea minimă a lui $x^2 = 0$

$x^2 \ge 0 \Rightarrow x^2 + 9 \ge 9 + 0$

$\Downarrow$

$\Rightarrow x^2 + 9 \ge 9$, $(\forall)\, x\in\mathbb{R}$

$\Rightarrow$ val. minimă 9 pt. $x = 0$

$F(x) = x^2 + 5$

$(\forall)\ x \in \mathbb{R} \Rightarrow x^2 \geq 0 \Rightarrow x^2 + 5 \geq 0 + 5 \Rightarrow x^2 + 5 \geq 5 \Rightarrow$

val. minimă $= 5$, pt. $x = 0$ | $a^2 \pm 2ab + b^2 = (a \pm b)^2$

$G(x) = x^2 + 6x + 12 = x^2 + 2 \cdot 3x + 3^2 + 3 = (x+3)^2 + 3$

$(\forall)\ x \in \mathbb{R} \Rightarrow (x+3)^2 \geq 0 \Rightarrow (x+3)^2 + 3 \geq 0 + 3 \Rightarrow (x+3)^2 + 3 \geq 3 \Rightarrow$

val. minimă este 3 şi se obţine dacă $x + 3 = 0 \Rightarrow x = -3$

$T(x) = x^2 + 10x + 31 = x^2 + 2 \cdot x \cdot 5 + 5^2 + 6 = (x+5)^2 + 6$

$(\forall)\ x \in \mathbb{R} \Rightarrow (x+5)^2 \geq 0 \Rightarrow (x+5)^2 + 6 \geq 0 + 6 \Rightarrow (x+5)^2 + 6 \geq 6$

$\Rightarrow$ val. minimă 6, se obţine pt. $x + 5 = 0 \Rightarrow x = -5$

$F(x) = x^2 - 12x + 40 = x^2 - 2 \cdot 6 \cdot x + 6^2 + 4 = (x-6)^2 + 4$

$(\forall)\ x \in \mathbb{R} \Rightarrow (x-6)^2 \geq 0 \Rightarrow (x-6)^2 + 4 \geq 0 + 4 \Rightarrow (x-6)^2 + 4 \geq 4 \Rightarrow$

val. minimă este 4, pt. $x - 6 = 0 \Rightarrow x = 6$

Determinaţi valoarea maximă a expresiilor $(x \in \mathbb{R}, y \in \mathbb{R})$

$E(x) = 10 - x^2$

$(\forall)\ x \in \mathbb{R} \Rightarrow x^2 \geq 0 \ |\cdot(-1) \Rightarrow -x^2 \leq 0 \Rightarrow +10 - x^2 \leq 0 + 10 \Rightarrow$

$\Rightarrow 10 - x^2 \leq 10$, $(\forall)\ x \in \mathbb{R} \Rightarrow$ val. maximă este 10, pt. $x = 0$.

$T(x) = 9 - y^2$

$(\forall)\ y \in \mathbb{R} \Rightarrow y^2 \geq 0 \ |\cdot(-1) \Rightarrow -y^2 \leq 0 \Rightarrow 9 - y^2 \leq 9 + 0 \Rightarrow 9 - y^2 \leq 9 \Rightarrow$

$\Rightarrow$ val. maximă 9 pt. $y = 0$ | $? - 9 = 12 \Rightarrow ? = 12 + 9 = 21$

$Q(x) = 12 - x^2 + 6x = 21 - x^2 + 2 \cdot 3x - 9 = 21 - (x^2 - 2 \cdot 3x + 3^2) \Rightarrow$

$Q(x) = 21 - (x-3)^2$

$(\forall)\ x \in \mathbb{R} \Rightarrow (x-3)^2 \geq 0 \ |\cdot(-1) \Rightarrow -(x-3)^2 \leq 0 \Rightarrow 21 - (x-3)^2 \leq 21 \Rightarrow$

val. maximă $= 21$ pt. $x - 3 = 0 \Rightarrow x = 3$

$F(x) = 15 - x^2 - 2x = 16 - x^2 - 2\cdot x\cdot 1 - 1^2 = 16 - (x^2 + 2\cdot x\cdot 1 + 1^2) \Rightarrow$

$F(x) = 16 - (x+1)^2$

$(\forall)\ x \in \mathbb{R} \Rightarrow (x+1)^2 \geq 0 \,|\cdot(-1) \Rightarrow -(x+1)^2 \leq 0 \Rightarrow 16 - (x+1)^2 \leq 16$

$\Rightarrow$ val. maximă 16 pt. $x+1=0 \Rightarrow x=-1$

Det. val. minimă:

$E(x;y) = \underline{x^2} + 4y^2 + \underline{10x} + 4y + 26$

$E(x;y) = x^2 + 2\cdot x\cdot 5 + \underbrace{5^2 + (2y)^2 + 2\cdot 2y\cdot 1 + 1^2}_{25+1=26} = (x+5)^2 + (2y+1)^2$

$(\forall)\ x;y \in \mathbb{R} \Rightarrow \left.\begin{array}{l}(x+5)^2 \geq 0\\(2y+1)^2 \geq 0\end{array}\right\} \Rightarrow \underline{(x+5)^2} + \underline{(2y+1)^2} \geq 0 \Rightarrow$

val. minimă 0 pt. $x+5=0 \Rightarrow x=-5$
$2y+1=0 \Rightarrow y=-\frac{1}{2}$

$a^2 + 2ab + b^2 = (a+b)^2$

$F(x,y) = x^2 + 18x + y^2 + 2y + 85$

$F(x,y) = x^2 + 2\cdot 9\cdot x + \underbrace{9^2 + y^2 + 2\cdot y\cdot 1 + 1^2}_{9^2+1^2+3=81+1+3=85} + 3 = (x+9)^2 + (y+1)^2 + 3$

$(\forall)\ x;y \in \mathbb{R} \Rightarrow \left.\begin{array}{l}(x+9)^2 \geq 0\\(y+1)^2 \geq 0\end{array}\right| \Rightarrow (x+9)^2 + (y+1)^2 + 3 \geq 3 \Rightarrow$ val. minimă $=3$ pt. $x=-9$ și $y=-1$

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